Introduction to separation of variables using the transport PDE

Prof W. D. Joyner, Math Dept, USNA

A *partial differential equation* (PDE) is an equation satisfied by an unknown function (called the dependent variable) and its partial derivatives. The variables you differentiate with respect to are called the independent variables. If there is only one independent variable then it is called an *ordinary differential equation.*

Examples include

- the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, where u is the dependent variable and x, y are the independent variables,
- the heat equation $u_t = \alpha u_{xx}$,
- and the wave equation $u_{tt} = c^2 u_{xx}$.

All these PDEs are of second order (you have to differentiate twice to express the equation). Here, we consider a first order PDE which arises in applications and use it to introduce the method of solution called *separation of variables.*

The transport or advection equation

Advection is the transport of a some conserved scalar quantity in a vector field. A good example is the transport of pollutants or silt in a river (the motion of the water carries these impurities downstream) or traffic flow.

The advection equation is the PDE governing the motion of a conserved quantity as it is advected by a given velocity field. The advection equation expressed mathematically is:

$$\frac{\partial u}{\partial t} + \nabla \cdot (u\mathbf{a}) = 0$$

where $\nabla\cdot$ is the divergence. Frequently, it is assumed that $\nabla \cdot \mathbf{a} = 0$ (this is expressed by saying that *the velocity field is solenoidal*). In this case, the above equation reduces to

$$\frac{\partial u}{\partial t} + \mathbf{a} \cdot \nabla u = 0.$$

Assume we have horizontal pipe in which water is flowing at a constant rate c in the positive x direction. Add some salt to this water and let $u(x,t)$ denote the concentration (say in lbs/gallon) at time t. Note that the amount of salt in an interval I of the pipe is $\int_I u(x,t)\,dx$. This concentration satisfies the *transport* (or *advection*) equation:

$$u_t + cu_x = 0.$$

(For a derivation of this, see for example Strauss [S], §1.3.) How do we solve this?

Solution 1: D'Alembert noticed that the directional derivative of $u(x,t)$ in the direction $\vec{v} = \frac{1}{\sqrt{1+c^2}}\langle c, 1\rangle$ is $D_{\vec{v}}(u) = \frac{1}{\sqrt{1+c^2}}(cu_x + u_t) = 0$. Therefore, $u(x,t)$ is constant along the lines in the direction of $\vec{v}$, and so $u(x,t) = f(x-ct)$, for some function f. We will not use this method of solution in the example below but it does help visualize the shape of the solution. For instance, imagine the plot of $z = f(x-ct)$ in (x,t,z) space. The contour lying above the line $x = ct+k$ (k fixed) is the line of constant height $z = f(k)$. $\square$

Solution 2: The method of *separation of variables* indicates that we start by assuming that $u(x,t)$ can be factored:

$$u(x,t) = X(x)T(t),$$

for some (unknown) functions X and T. (One can shall work on removing this assumption later.) Substituting this into the PDE gives"

$$X(x)T'(t) + cX'(x)T(t) = 0.$$

Now separate all the x's on one side and the t's on the other (divide by $X(x)T(t)$):

$$\frac{T'(t)}{T(t)} = -c\frac{X'(x)}{X(x)}.$$

It is impossible for a function of an independent variable x to be identically equal to a function of an independent variable t unless both are constant. So, we have two ODEs:

$$\frac{T'(t)}{T(t)} = K, \quad \frac{X'(x)}{X(x)} = -K/c,$$

for some (unknown) constant K. Solving, we get

$$T(t) = c_1 e^{Kt}, \quad X(x) = c_2 e^{-Kx/c},$$

so, $u(x,t) = Ae^{Kt-Kx/c} = Ae^{-\frac{K}{c}(x-ct)}$, for some constants K and A (where A is shorthand for $c_1 c_2$; in terms of D'Alembert's solution, $f(y) = Ae^{-\frac{K}{c}(y)}$). The "general solution" is a sum of these (for various A's and K's). $\Box$

Example: Assume water is flowing along a horizontal pipe at 3 gal/min in the x direction and that there is an initial concentration of salt distributed in the water with concentration of $u(x,0) = e^{-x}$. Using separation of variables, find the concentration at time t. Plot this for various values of t.

Solution: The method of separation of variables gives the "separated form" of the solution to the transport PDE as $u(x,t) = Ae^{Kt-Kx/c}$, where $c = 3$. The initial condition implies

$$e^{-x} = u(x,0) = Ae^{K\cdot 0 - Kx/c} = Ae^{-Kx/3},$$

so $A = 1$ and $K = 3$. Therefore, $u(x,t) = e^{3t-x}$.

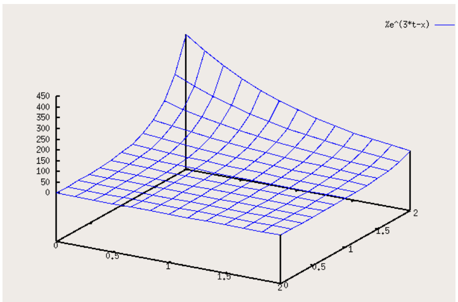

Figure 1: Transport with velocity $c = 3$.

$\Box$

What if the initial concentration was not $u(x,0) = e^{-x}$ but instead $u(x,0) = e^{-x} + 3e^{-5x}$? How does the solution to

$$u_t + 3u_x = 0, \quad u(x,0) = e^{-x} + 3e^{-5x}, \tag{1}$$

differ from the method of solution used above? In this case, we must use the fact that (by superposition) "the general solution" is of the form

$$u(x,t) = A_1 e^{K_1(t-x/3)} + A_2 e^{K_2(t-x/3)} + A_3 e^{K_3(t-x/3)} + \ldots\,, \tag{2}$$

for some constants $A_1, K_1, ...$ To solve this PDE (1), we must answer the following questions: (1) How many terms from (2) are needed? (2) What are the constants $A_1, K_1, ...$? There are two terms in $u(x, 0)$, so we can hope that we only need to use two terms and solve

$$e^{-x} + 3e^{-5x} = u(x, 0) = A_1 e^{K_1(0-x/3)} + A_2 e^{K_2(0-x/3)}$$

for A_1, K_1, A_2, K_2. Indeed, this is possible to solve: $A_1 = 1$, $K_1 = 3$, $A_2 = 3$, $K_1 = 15$. This gives

$$u(x, t) = e^{3(t-x/3)} + 3e^{15(t-x/3)}.$$

References

[A] Advection, `http://en.wikipedia.org/wiki/Advection`

`http://en.wikipedia.org/wiki/Advection_equation`

[S] W. Strauss, *Partial differential equations, an introduction*, John Wiley, 1992.